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Freedom, Opportunity and Wellbeing

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Abstract

This paper reexamines key results from the measurement of opportunity freedom, or the extent to which a set of options offers a decision maker real opportunities to achieve. Three cases are investigated: no preferences, a single preference, and plural preferences. The three corresponding evaluation methods – the cardinality relation, the indirect utility relation, and the effective freedom relation – and their variations are considered within a common axiomatic framework. Special attention is given to representations of freedom rankings, with the goal of providing practical approaches for measuring opportunity freedom and the extent of capabilities.

Keywords: freedom, individual choices, welfare, capabilities, axiomatic approach, orderings.

JEL classification: D63, D03, Z13, I31.

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The Oxford Poverty and Human Development Initiative (OPHI) is a research centre within the Oxford Department of International Development, Queen Elizabeth House, at the University of Oxford. Led by Sabina Alkire, OPHI aspires to build and advance a more systematic methodological and economic framework for reducing multidimensional poverty, grounded in people's experiences and values.

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I. Introduction

Choice is a central feature of economic models and the evaluation of choice outcomes is fundamental to welfare economics and to the development of policies for improving welfare. Traditional evaluation methods focus purely on the outcome – say, the bundle of goods or the occupation – selected by the decision maker. Yet a compelling case can be made for broadening the basis of evaluation to include other factors such as the *process* by which the choice was made and the number and characteristics of *unchosen* alternatives (Sen 1988, 2002). This is the fundamental concern of the *freedom* approach to welfare economics, which argues that broader contextual features of choice may have normative value beyond their impact on the specific alternative selected.

The relevant issues can be illustrated by the case of Jill, a newly minted graduate who has just chosen teaching as her life's vocation. An evaluation of Jill's life chances and wellbeing might consider the salary, benefits and other characteristics of this profession, and the lifestyle that it might entail. However, suppose that it were revealed that the process by which Jill "chose" her occupation included a preliminary stage in which most other occupations, including the one she preferred – civil engineering – were simply removed from her choice set by a society that regarded them as inappropriate for women. Or alternatively, suppose her home was a small village and significant pressure was brought to bear on her to adhere to the expressed preferences of, say, a local chieftain. Or suppose that the actual choice was made on her behalf by some nameless government functionary. It seems reasonable that an evaluation of the choice outcome should not be indifferent to the process that led to that outcome – particularly when the process can stray so far from the idealized, traditional view of choice.

All of these scenarios involve a compromised choice process – the first in which the option set is artificially limited (agenda manipulation), the second in which the chooser's preferences are distorted (preference manipulation), and the final in which the selection is made by another person (choice manipulation). Each likely delivers an outcome that is different to one that would arise in the absence of manipulation. But even if the selected option – Jill's vocation in the above scenarios – were identical to the one obtained under autonomous, free choice, the fact that the process is deeply flawed may well change our evaluation of the outcome. The low level of personal autonomy and the clear vulnerability to luck and the whims of others detract from an otherwise agreeable conclusion to the decision process.

Sen (2002) has discussed at some length the "process aspect of freedom" whose roots are found in many classic works in economics and philosophy (Kant 1788, Smith 1759, Mill 1848, 1859). Contemporary works in economics and related fields have stressed the importance of process freedoms (Agarwal, Humphries and Robeyns 2007, Frank 1988, Nagel 1970, Nozick 1974, Rawls 1971, Scitovsky 1986, Sugden 1981, 1998, van Hees 1998). There are many contemporary analyses of process freedom, including empirical efforts to evaluate the extent to which actions are motivated by external manipulations (Alkire 2008). Additional applications to the study of corruption, competition policy, law and economics, and behavioral economics can be readily envisioned.

This paper focuses on a second way that freedom enters into the analysis of choice and welfare that is called the "opportunity aspect of freedom" by Sen (2002). Suppose that the agent has free choice from a naturally occurring set of options (and hence unrestricted process freedom). A larger set of options offers the decision maker a greater opportunity to obtain a more favorable selection.[1] The usual method of evaluation, however, ignores all available options apart from the chosen one. In the case of Jill, this would amount to saying that a setting in which her *only* option available is to become a civil engineer (her

[1] This presumes that the options being added have positive value; and that the additional costs of decision making with more options are not so large as to outweigh the benefits.

favorite choice) is equivalent to the setting in which she has before her a broad array of plausible and desirable occupations, from which she selects civil engineering. In contrast to this standard view, the wider range of real opportunities is seen to enhance the opportunity aspect of freedom even when the final choice is the same.

Much of the theoretical literature in opportunity freedom has been concerned with identifying a source of value for unchosen options, and providing a way of measuring this value. One approach posits that there is an intrinsic value to having a greater number of options, irrespective of the qualities the extra options bring. This naturally leads to an emphasis on the *quantity* of options and hence the counting method for evaluating opportunity freedom presented by Pattanaik and Xu (1990). Alternatively, one can consider the *quality* of alternatives as indicated by the preferences or utility of the decision maker, and evaluate opportunity sets by the value of the best element – known as the indirect utility approach. There are many plausible ways of incorporating quantity and quality elements to measure the opportunity aspect of freedom and we present several below.

There is yet another route by which unchosen options are seen to be valuable. Let us return to example of Jill at the moment of decision, and ask: If the preference relation were fixed and known, so that the choice could be predicted with certainty, why would the other options in the opportunity set have any value at all? A traditional indirect utility view would conclude that there is no additional value. However, this might not be the case if, when we approached Jill, she were of two minds about her preferences and choice. If Jill thought that she *might* prefer teaching over civil engineering, but also might have the *opposite* preferences, then there may be value to having additional options due to Jill's uncertainty regarding her preferences. The situation is analogous to an analysis of flexibility by Kreps (1979) in which the value of having more options originates from the presence of unresolved uncertainty, and is also similar to Weisbrod's (1964) notion of "option value." It is the motivation of the final, "plural preference" approach to opportunity freedom given below.

The most important exponent of the analysis of freedom and choice has been Amartya Sen, who through a series of books and papers has argued forcefully for broadening the informational bases of welfare economics to include the opportunity freedoms we have. He defines *capabilities* to be the real freedoms people have to achieve beings and doings they value and have reason to value (Sen 1980, 1985, 1992). The *capability set* is "a set of vectors of functionings, reflecting the person's freedom to lead one type of life or another... to choose from possible livings" (Sen 1992 p 40).[2] The capability approach evaluates social and economic outcomes in the space of human lives and yet maintains a substantive role for opportunity freedom. The main challenge is how to measure the extent of choice in a given capability set practically, or how to measure opportunity freedom. The work presented below can be viewed as providing the theoretical basis for empirical measures of capabilities.

One noteworthy feature of the capability approach is that it does not reduce wellbeing into a single dimension (such as income or utility) but instead is inherently multidimensional. For example, a person's capability set might include capabilities related to nutrition, to knowledge, to friendship, or other dimensions. The capability approach would suggest that an adequate measure of wellbeing (or poverty) must reflect the conditions in multiple dimensions simultaneously. For example, the Human Development Index, to which Sen contributed (UNDP 1990), measures wellbeing with respect to health, education, and income. More recently, the capability approach has provided the conceptual framework for a substantial literature on multidimensional poverty measurement (Anand and Sen 1997, Atkinson

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[2] See also the survey of Basu and Lopez-Calva (2010). Sen's 2009 book *The Idea of Justice* restates the importance of considering human advantage in the space of capabilities.

2003, Bourguignon and Chakravarty 2003, Alkire and Foster 2007) designed to supplement or replace traditional income based poverty measures.[3]

This paper reviews the literature on the evaluation of *opportunity freedom*, intuitively defined as the extent to which a set of options, or an *opportunity set*, offers an agent a range of meaningful choices.[4] The basic objects of study are *rankings* of opportunity sets, *measures* of freedom, and the *axioms* they satisfy. We survey some of the main approaches to constructing freedom rankings and explore combinations of axioms that characterize specific freedom rankings. We reinterpret several axiomatic characterizations of rankings as representation theorems. Since many freedom rankings are incomplete, we often use vector-valued functions to represent the underlying ranking, with non-comparable vectors indicating situations where two sets cannot be ranked.[5] We also explore extensions of the resulting quasiorderings to complete orderings and note the additive structure underlying each measure of freedom.

A key tension in the measurement of freedom arises between the *quantity* and *quality* of available alternatives, and different evaluation strategies emphasize one of these aspects exclusively or attempt to include them both. Of critical importance is whether the agent has a preference ordering over *alternatives* that may be invoked in constructing a freedom ranking over *sets*. Three apparently distinct cases will be investigated: (i) where there is no preference ordering, (ii) where there is a single preference ordering, and (iii) where there is a collection of potential orderings. In the first case, the absence of preference information means that a freedom measure can only reflect quantitative aspects of opportunity sets. This can lead to the most basic kind of freedom ranking – set inclusion – or, under additional axioms (e.g., Pattanaik and Xu, 1990) to a complete ordering where freedom is measured by the cardinality of a given set. The severe incompleteness of the former and the profound arbitrariness of the latter provide a strong incentive to incorporate additional information on the relative desirability of alternatives.

In the second case, where a preference relation or a utility function over individual alternatives is provided, the quality of the elements in a given set can be incorporated into the measurement of freedom. The traditional economic model of decision making leads us to identify the value of a given set with the utility of its best element. The resulting *indirect utility* criterion for measuring freedom ignores the number and characteristics of unchosen alternatives in the set, instead viewing greater freedom purely in terms of the ability to achieve a better outcome. It can be axiomatically characterized and has an intuitive representation in terms of the size of the set of alternatives the best element dominates.

The indirect utility freedom measure, like the cardinality measures of freedom, generates a complete ordering of opportunity sets by focusing exclusively on one aspect of freedom (in this case quality instead of quantity). If both aspects are deemed to have salience, new rankings can be constructed by either *aggregating* the cardinal and indirect utility measures of freedom or by taking their *intersection* wherein an increase in freedom requires an increase in both. We present these approaches and explore the properties of the resulting combination freedom rankings.

[3] For example, see Sen (1976), Foster, Greer, and Thorbecke (1984, 2010), and Foster and Sen (1997).

[4] This is by no means an exhaustive account of the existing literature on freedom evaluation. Instead, the goal here is to highlight some of the key results, and several new findings, within a coherent overarching framework. More extensive presentations can be found in the surveys of Barbera, Bossert, and Pattanaik (2004) and Dowding and van Hees (2009) as well as in collections such as Laslier et al (1998).

[5] See Putnam (1986) for an illuminating discussion of incompleteness. Vector valued representations of incomplete relations abound in economics, and include Pareto dominance, mean variance analysis, stochastic dominance, among other well known examples. See Foster (1993) or Ok (2002).

The indirect utility approach depends entirely on the utility of a best alternative in a set and ignores the unchosen alternatives in the set and the unavailable alternatives outside the set. One could augment this informational basis by including information on the second best utility level, the third and so forth, resulting in many dimensions of quality. Once again the multidimensionality might be dealt with by aggregation (yielding a freedom measure and a complete ordering) or by intersection (yielding a quasiordering with a vector representation). The resulting freedom rankings and their properties are also discussed below.

In the third *plural preferences* case, the agent has a *set* of potential preference relations, each of which may be the agent's actual preference when it comes time to select an alternative from the opportunity set. However, the agent must rank sets *before* it is known which preference will obtain. A natural way of dealing with this uncertainty is to rank sets according to the intersection of the indirect utility orderings of each potential preference. The resulting *effective freedom* ranking judges one set to have greater than or equal freedom to another if all indirect utility orderings agree this is so. As with the indirect utility approach, it is the quality rather than the number of alternatives in a set, that determines its ranking. If all preferences are very similar, then one very good alternative may do the trick and the freedom ranking looks similar to the indirect utility ranking; but if preferences vary dramatically, then a variety of alternatives may be needed in order for a given set to achieve greater freedom. This is the sense in which the multiple preference approach can favor greater numbers of alternatives. We present the effective freedom approach and examine the axioms it satisfies. We show interesting representations of the quasiordering, including a vector valued representation and provide an interesting extension of the quasiordering to a complete ordering based on a count of options in the associated effective opportunity set. As the set of potential preferences varies from a single preference ordering to all possible orderings, the effective freedom relation varies from the indirect utility ordering to set inclusion, while the counting ordering varies from the indirect utility ordering to the cardinal ordering of Pattanaik and Xu (1990).

Sen (2002) considers a subrelation of the effective freedom quasiordering in which the order of intersecting and evaluating indirect utility are reversed: first the preferences are intersected to obtain a quasiordering of alternatives which then is used to compare sets. The altered order ensures that Sen's freedom relation is generally less complete than the effective freedom quasiordering; however, when the Sen relation applies, it conveys additional information that is especially useful in a model where the timing of events is also reordered. We present the Arrow (1995) extension of the effective freedom quasiordering to a complete ordering, which assigns probabilities to the various preferences – now given as utility functions – and aggregates over indirect utility levels. We note that when all logically possible preference orderings are possible and is equally likely, Arrow's extension can reduce to Pattanaik and Xu's cardinality measure once again.

II. Notation

In what follows, the (finite) universal set of n alternatives will be denoted by X. An *opportunity set* or *menu* is any nonempty subset of X, with the set of all such menus being denoted by Z. Freedom is evaluated using a binary relation R on Z, with A R B indicating that the level of freedom offered by $A \sum Z$ is as great as the level offered by $B \sum Z$. R is typically assumed to be a *quasiordering*, which means that R is reflexive and transitive. Notice that completeness (requiring the relation to rank any two opportunity sets) is not being assumed; it is entirely possible that neither A R A' nor A' R A holds for any given pair A and A' in Z.

The asymmetric part of R, denoted P, and the symmetric part of R, denoted I, decompose this relation in such a way that A P B indicates that A has strictly more freedom than B, while A I B ensures that the two opportunity sets have the same level of freedom. It is easily shown that both I and P are both transitive relations. The cardinality of menu A, denoted $|A|$, will play an important role in the first approach to measuring freedom. The *cardinality freedom ranking* R^C is defined by A R^C B if and only $|A| \geq |B|$; it judges freedom on the basis of the number of elements in a menu. The second approach will make use of a preference ordering R_1 on X, whose ranking over individual alternatives will be used in constructing the freedom ranking R over menus of alternatives. The *indirect utility freedom ranking* R^U is defined by A R^U B if and only if there exists $x \sum A$ such that $x R_1 y$ for all $y \sum B$; it judges freedom on the basis of the best alternative(s) in the menus. The third approach constructs R with the help of a finite number m of preference orderings $R_1,\ldots,R_m$ on X; the collection of preference orderings is denoted by $\mathfrak{R} = \{R_1,\ldots,R_m\}$.

III. Freedom counts

Our first approach to measuring freedom follows the old saw that certain academic officials can count but can't read; in other words, quantity is observable, but quality cannot be easily ascertained. Regardless of the accuracy of this observation, it provides a handy place to begin our discussion of the measurement of freedom viewed as the number of available choices. Our focus is the approach of Pattanaik and Xu (1990), whose axiomatic result characterizing the "cardinality" measure of freedom has produced much discussion. We provide an intuitive proof of the characterization theorem, and then subject it to a sensitivity analysis in which the axioms are progressively relaxed until the essential source of the result is revealed. We also translate the theorem into a vector space structure, to show that it has the form of a traditional additive representation theorem.

III.1 Rankings and Axioms

To frame their analysis, Pattanaik and Xu (1990) take as their primary concern the *intrinsic* value of freedom, not its instrumental use in achieving utility or some other end. They provide a series of axioms to help define their notion of freedom. The first of these axioms requires the freedom ranking to have a rather strong form of consistency: that the relative ranking of a pair of opportunity sets is unchanged when an additional alternative is added to, or when a common alternative is removed from, both sets.

> <u>Simple Independence</u>. Let A, B, and C be elements of Z for which
>
> $A \cap C = \varnothing = B \cap C$ and $|C| = 1$. Then A R B if and only if $(A \cup C)$ R $(B \cup C)$.

Notice that this axiom actually comprises two separate conditional requirements. The first requires that if A is comparable to B by the freedom ranking, then the enlarged set $(A \cup C)$ is *comparable* to the enlarged set $(B \cup C)$, and moreover, the ranking of these sets should be the *same* as the ranking of A versus B. The second requires that if $(A \cup C)$ can be compared to $(B \cup C)$ then the smaller sets A and B must be comparable and have a ranking that corresponds to the ranking over the larger sets. The latter requirement can be problematic for certain notions of freedom. In particular, if A and B are initially non comparable, or one is strictly preferred to the other, but both are dominated by a third set C, then we might expect $(A \cup C)$ to be indifferent to $(B \cup C)$. But this is not allowed under the second part of this axiom.

One natural requirement for a freedom ranking is that the addition of an alternative to a given opportunity set should either enhance freedom or at least leave it unchanged. Pattanaik and Xu focus on the special case where the initial set under consideration is singleton and then strengthen the requirement to demand a *strict* increase in freedom when *any* given new alternative becomes available.[6]

Simple Strict Monotonicity Let A and B be elements of Z with $|A| = 1$ and

$|B| = 2$. If $A \subset B$ then B P A.

Note that under this version of monotonicity, indifference between a two-element menu and a singleton subset is ruled out – regardless of the characteristics or quality of the extra alternative.

The final axiom assumes that all singleton sets are comparable and specifies a particular ranking for them.[7]

Simple Anonymity Let A and B be elements of Z with $|A| = 1$. If $|A| = |B|$

then A I B.

This axiom ensures that the singleton opportunity sets all have the same level of freedom, irrespective of the characteristics or quality of the alternative each happens to contain.

Now which freedom rankings satisfy all three of these properties, as well as the maintained assumption that R is a quasiordering? Pattanaik and Xu provide the answer as follows:

Theorem 1 R is a quasiordering satisfying simple independence, simple strict monotonicity and simple anonymity, if and only if R is the cardinality ordering R^C.

Thus, the only quasiordering consistent with this collection of properties is the one that is based on a simple counting of alternatives in the respective sets.

We will presently provide an additional interpretation of this theorem, but first let us explore some direct implications of the Pattanaik and Xu axioms. Notice that the three axioms are stated with restrictions on the sizes of sets (hence the term "simple"). Consider the following counterparts in which these restrictions have been removed.[8]

Independence. Let A, B, and C be elements of Z for which

$A \cap C = \varnothing = B \cap C$. Then A R B if and only if $(A \cup C)$ R $(B \cup C)$.

Strict Monotonicity Let A and B be elements of Z. If $A \subset B$ then B P A.

Anonymity Let A and B be elements of Z. If $|A| = |B|$ then A I B.

[6] Sen (2002, p. 686) calls this the Superiority of Some Choice axiom.

[7] This is called the Principle of No Choice by Sen (2002).

[8] Sen's (2002, pp. 687-693) terminology for these axioms is Suppes Additivity, Strict Set Dominance, and Equi-cardinality Indifference, respectively.

The next result shows that this collection of axioms is entirely as general as the three simple counterparts.[9]

> Lemma Let R be a quasiordering. Then independence, strict monotonicity and anonymity are equivalent to simple independence, simple strict independence and simple anonymity.

Proof If R satisfies the unrestricted version of the three axioms, then it clearly satisfies the simple versions. Conversely, suppose that R satisfies the three axioms of Pattanaik and Xu. It is immediate that R must satisfy independence. To see this, let $A \cap C = \varnothing = B \cap C$ where C contains more than one alternative. Rather than adding all the elements of C at once to A and B, we can do this one element of C at a time to obtain a series of pairs of sets beginning with A and B and ending with $A \cup C$ and $B \cup C$. Applying simple independence to adjacent pairs in this series yields a string of equivalences from A R B to $(A \cup C)$ R $(B \cup C)$, and hence independence.

Next we show that R satisfies strict monotonicity. Suppose that $A \supset B$ where $|A| - |B| = 1$. Select any subset C of B such that $B' = B \setminus C$ has cardinality $|B'| = 1$ and $A' = A \setminus C$ has cardinality $|A'| = 2$. Then by simple strict monotonicity it follows that A' P B' and hence by independence we have A P B. Now let $k > 1$ and suppose that $A \supset B$ entails A P B for all A and B satisfying $|A| - |B| < k$. We will show that this is also true for $|A| - |B| = k$. Let $A \supset B$ with $|A| - |B| = k$, and pick any B" such that $A \supset B''$ and $B'' \supset B$. Then by the induction hypothesis A P B" and B" P B, and hence we obtain A P B by the transitivity of P. This establishes strict monotonicity.

Finally, to show that R satisfies anonymity, let $k > 1$ and suppose that $|A| = |B|$ entails A I B for all A and B satisfying $|A| < k$. We will show this is also true for $|A| = k$. So let $|A| = |B| = k$. Pick c in $A \cap B$ if this set is nonempty and denote $A' = A \setminus \{c\}$ and $B' = B \setminus \{c\}$. By independence and the induction hypothesis we have A I B. If $A \cap B = \varnothing$, then pick any a in A and b in B, and denote $A' = A \setminus \{a\}$ and $B' = B \setminus \{b\}$. By the induction hypothesis, we have A' I B' and hence $(A' \cup \{a\})$ I $(B' \cup \{a\})$ by independence and moreover $(B' \cup \{a\})$ I $(B' \cup \{b\})$ by simple anonymity and independence. Hence, by the transitivity of I, we have A I B. This verifies that anonymity holds.

This result shows that the unrestricted versions of the axioms are collectively equivalent to the original simple versions. Armed with this observation, the proof of the Pattanaik and Xu theorem is immediate.

Proof (Theorem 1): It is clear that the cardinality ordering R^C satisfies all three axioms. Now let us begin with any quasiordering R satisfying all three axioms. We want to show that R is R^C. Suppose that A R^C B or, equivalently, $|A| \geq |B|$. Pick any $C \subseteq A$ with $|C| = |B|$. By strict monotonicity (or reflexivity) we know that A R C, while anonymity ensures that C I B. Hence by reflexivity we conclude that A R B. Conversely, suppose that it is not the case that A R^C B or, equivalently, that we have $|A| < |B|$. Select $C \supset A$ with $|C| = |B|$. By anonymity we have B I C while by strict monotonicity we know C P A. By transitivity we cannot have A R B. Therefore, A R^C B if and only if A R B.

The simple anonymity axiom is transformed by the independence axiom into anonymity, which in turn implies that the indifference class associated with a given opportunity set includes all sets having the same number of alternatives. By independence, the characteristics of R are dependent on how it compares opportunity sets with cardinality 1 and 2. What role does simple strict monotonicity play in proof of the characterization result? First, it ensures that there is *comparability* across opportunity sets of sizes 1 and 2. This fact alone (along with independence) leaves open three possibilities for R: either it is

9 Related results are found in Sen (2002, T.11.3).

represented by $|A|$, it is represented by $-|A|$, or it is constant across all opportunity sets A. The second role that simple strict monotonicity plays is to determine the particular *orientation* for this ranking, thus ensuring that the first of these possibilities – the cardinality ordering – is the only one that applies.

We noted above that the strict forms of monotonicity are not unambiguously acceptable, since they require that the addition of an alternative must lead to a strictly higher level of freedom, irrespective of the alternative's characteristics. On the other hand, few would argue that the addition of an alternative would not lead to a weakly higher (higher or the same) level of freedom. The following weaker monotonicity axioms allow freedom to remain the same when a new alternative is added, and hence are arguably more natural requirements for a freedom ranking.[10]

Simple Monotonicity Let A and B be elements of Z with $|A| = 1$ and $|B| = 2$. If $A \subseteq B$ then B R A.

Monotonicity Let A and B be elements of Z. If $A \subseteq B$ then B R A.

Each of these two axioms is implied by its strict counterpart. Moreover, simple monotonicity and monotonicity are clearly equivalent for a quasiordering R satisfying independence. To see the marginal importance of the strictness of the monotonicity axiom, we explore what happens to the Pattanaik and Xu result when the strict form of monotonicity is replaced by the weaker version. Consider the *trivial* freedom ranking R^T defined by A R^T B for all A, B Σ Z, so that R^T regards all opportunity sets as having the same level of freedom. It turns out that there are *two* freedom rankings consistent with the previous two axioms and the less restrictive monotonicity axiom.

Theorem 2 R is a quasiordering satisfying simple independence, simple monotonicity and simple anonymity if and only if R is the cardinal ordering R^C or the trivial ordering R^T.

Proof It is clear that the orderings R^C and R^T satisfy the three axioms. Now let us suppose that R is any quasiordering satisfying the three. We want to show that R is either R^C or R^T. So let x and y be any two distinct alternatives in X. By simple monotonicity, we obtain {x, y} R {y}. Suppose first that {x, y} P {y} for this particular x and y. Then by anonymity and transitivity it follows that A P B for any A and B in Z with $|A| = 2$ and $|B| = 1$. Hence R must satisfy simple strict monotonicity and so by Theorem 1 we know that R is R^C. Alternatively, suppose that {x, y} I {y} for the given x and y. Then by Anonymity and transitivity we obtain A I B for any A and B in Z with $|A| = 2$ and $|B| = 1$. Independence extends this conclusion to any A and B with $|A| - |B| = 1$, and hence, by transitivity, to all A and B in Z. Consequently, R is R^T.

This result offers strong evidence that strict monotonicity is not the driving force behind the Pattanaik and Xu result. Even when simple strict monotonicity is replaced by simple monotonicity – an innocuous requirement that augmenting a singleton set by an additional alternative either increases freedom or leaves it unchanged – the simple independence and simple anonymity axioms conspire to ensure that the resulting ordering is either the cardinal ordering or the trivial ordering (that considers all opportunity sets to have the same level of freedom). The "strictness" of the monotonicity in the Pattanaik and Xu theorem only serves to remove the trivial ranking.

As mentioned above, the independence axioms are made up of two parts: an implication from smaller sets (A R B) to larger sets ((A ∪ C) R (B ∪ C)); and a converse implication from the larger sets to the smaller. The first of these is fundamental to the notion of freedom: adding the same alternatives to two

[10] Monotonicity is called Weak Set Dominance in Sen (2002).

ranked sets cannot render the sets unrankable, nor can it strictly reverse the ranking. The second component, by which removing the same alternatives from the two sets preserves the ranking, is quite restrictive and, indeed, is inconsistent with many plausible conceptions of freedom. We now explore the implications of removing the objectionable part of this axiom. Consider the following requirements for freedom rankings:

> Simple Semi-Independence. Let A, B, and C be elements of Z for which
>
> $A \cap C = \varnothing = B \cap C$ and $|C| = 1$. Then A R B implies $(A \cup C)$ R $(B \cup C)$.
>
> Semi-Independence. Let A, B, and C be elements of Z for which
>
> $A \cap C = \varnothing = B \cap C$. Then A R B implies $(A \cup C)$ R $(B \cup C)$.

These axioms retain the unambiguous part of the original independence axioms, which requires the addition of elements to A and B not to disrupt the original ranking of A and B. It is easy to show by a proof entirely analogous to the one given above for the independence axioms that the two forms of semi-independence are equivalent. Moreover, this more limited form of independence is all that is needed to show that simple monotonicity is equivalent to monotonicity and simple anonymity is equivalent to anonymity.

Now what is the impact of relaxing simple independence in Theorem 2 to simple semi-independence? For k = 1,...,n, define the *censored cardinality measure* $|A|^k$ by: $|A|^k = |A|$ for $|A| < k$ and $|A|^k = k$ for $|A| \geq k$. Consider the *censored freedom ranking* R^k defined by A R^k B if and only if $|A|^k \geq |B|^k$. Notice that R^n is simply R^C, the cardinality ranking, which measures freedom in terms of the number of elements in the set; while R^1 is R^T, the trivial ranking for which all sets have equal freedom. For k between 1 and n the freedom ranking R^k orders sets on the basis of their cardinality censored at k. Freedom is augmented by an additional alternative so long as the total number of alternatives remains below k; thereafter, adding an alternative has no effect on freedom. We have the following result.

> Theorem 3 R is a quasiordering satisfying simple semi-independence, simple monotonicity and simple anonymity if and only if R is R^k for some k = 1,…,n.

Proof It is obvious that each R^k is a quasiordering that satisfies simple monotonicity and simple anonymity. Moreover, suppose that A, B, and C are elements of Z for which $A \cap C = \varnothing = B \cap C$ and $|C| = 1$, and suppose that A R^k B, or equivalently, $|A|^k \geq |B|^k$. There are three possibilities. First, if $|A| < k$, then $|A \cup C|^k = |A| + 1 \geq |B| + 1 = |B \cup C|^k$ and hence $(A \cup C)$ R^k $(B \cup C)$. Second, if $|A| \geq k > |B|$, then $|A \cup C|^k = |A| \geq |B| + 1 = |B \cup C|^k$ and hence $(A \cup C)$ R^k $(B \cup C)$. Third, if if $|B| \geq k$, then $|A \cup C|^k = |A| = |B| = |B \cup C|^k$ and hence $(A \cup C)$ R^k $(B \cup C)$. Thus, R^k satisfies simple semi-independence.

Now, suppose that R is any quasiordering satisfying simple semi-independence, simple monotonicity and simple anonymity. By the above discussion, R satisfies semi-independence, monotonicity, and anonymity. Monotonicity ensures that all sets can be compared and that $|A| \geq |B|$ implies that A R B. Pick a menu C with the lowest cardinality such that X I C, i.e., C has maximal freedom. Let $k = |C|$ and note that for any sets A and B in Z with $|A|^k = |B|^k$ we must have A I B. Further, it is clear by definition of C that any set, say C', satisfying $|C| = |C'| + 1$ would have to satisfy C P C'. Indeed, any pair of sets D' and D satisfying $k > |D| = |D'| + 1$ would have to satisfy D P D'; otherwise, if D I D' then semi-independence, anonymity and transitivity would imply that C I C', contrary to the above. Hence, for any A and B in Z with $|A|^k > |B|^k$ we must have A P B. It follows immediately that R is R^k.

Theorem 3 provides convincing evidence that simple anonymity is the key axiom underlying the Pattanaik and Xu result. Even in the presence of the unambiguous simple semi-independence axiom (which requires that adding a new alternative to a weakly ranked pair of menus preserves the ranking) and the equally justifiable simple monotonicity axiom (which ensures that adding a new alternative to a singleton set results in set with greater or equal freedom), the simple anonymity axiom restricts consideration to the family of censored cardinal freedom rankings R^k, including the cardinal ordering R^C and the trivial ordering R^T. Strengthening the monotonicity axiom removes the trivial freedom ordering from consideration, but leaves all R^k with $k > 1$; strengthening the independence axiom removes the intermediate censored cardinal rankings with $1 < k < n$, leaving the two extremes of R^T and R^C. Simple anonymity along with both of the more demanding axioms then reduces the possibilities to the cardinality ranking R^C.

III.2. Additive Representations and Cardinal Freedom

The theorem of Pattanaik and Xu not only offers a characterization of a freedom ranking, it provides a function that represents the relation (namely $|A|$). This view of their theorem as a representation theorem has not been especially emphasized in the literature. However, this perspective reveals that their theorem is quite closely related to traditional results on additive representations, which in turn suggests other theorems, including an extension of the theorem to "fuzzy" opportunity sets. We now present this reinterpretation and its potential implications.

Select an arbitrary enumeration of the n elements of X. Every opportunity set in Z can be represented by a vector v from the set $V = \{v \in R^n \setminus \{0\}: v_i = 0 \text{ or } v_i = 1\}$, where $v_i = 1$ indicates that the ith element of X is present in the given set while $v_i = 0$ indicates that the ith element is absent. So, for example, given a three element set X with elements enumerated as a,b,c, the vector v = (1,0,1) corresponds to the set {a,c}. With this reinterpretation of sets as vectors, each of the above axioms can be directly translated to the new environment. In what follows e_i denotes the ith basis vector, and v, w, v', and w' are elements of V.

<u>Quasiordering</u> The relation R is reflexive and transitive.

<u>Simple Anonymity</u> e_i I e_k for all i, k.

<u>Simple Strict Monotonicity</u> $(e_i + e_k)$ R e_i for all $i \neq k$.

<u>Simple Independence</u> Let $k \in \{1,\ldots,n\}$. Suppose that $v_i = v'_i$ and $w_i = w'_i$ for all $i \neq k$, while $v_k = w_k$ and $v'_k = w'_k$. Then v R w if and only if v' R w'.

<u>Anonymity</u> v I (Πv) for all v and every nxn permutation matrix Π

<u>Strict Monotonicity</u> $v > w$ implies v P w.

<u>Independence</u> Let $K \subset \{1,\ldots,n\}$. Suppose that $v_i = v'_i$ and $w_i = w'_i$ for all $i \notin K$, while $v_k = w_k$ and $v'_k = w'_k$ for all $k \in K$. Then v R w if and only if v' R w'.

It is an easy exercise to verify that each axiom corresponds exactly to its twin given above in the contexts of sets. However, the properties are now much more familiar. For example, simple anonymity becomes a requirement that any two usual basis vectors are ranked identically by the relation, while full anonymity requires permutations of any initial vector to likewise leave the ranking unchanged. Strict monotonicity is the standard version that requires an increase in one dimension, and no decrease in any other, to lead to a strict increase in the ranking; the simple version requires the sum of any two basis vectors to be ranked strictly higher than either of them. Independence is indeed a standard separability axiom used in additive representations, while simple independence is a more limited form of the same.[11]

It is an easy matter also to see that the Pattanaik and Xu result is equivalent to the following additive representation theorem on the set V.

> Theorem 4 Let R be any quasiordering on the set V satisfying simple anonymity, simple strict monotonicity, and simple independence. Then v R v' if and only if $\sum_{i=1}^{n} \mathbf{v_i} \geq \sum_{i=1}^{n} \mathbf{v_i}'$.

In other words, the three axioms are sufficient to ensure that the quasiordering R has an unweighted *additive representation*.

One notable aspect of this result is that the completeness of R is not being assumed a priori, but rather falls out as an implication of the other axioms. But if we knew a priori that R were a complete ordering, the theorem would seem very natural indeed – even intuitive. The independence requirement is entirely analogous to traditional separability. Combined with strict monotonicity – not an unusual property for an ordering – we could well expect there to be an additive representation $\Sigma_i f_i(v_i)$ for R with increasing component functions f_i. Anonymity then acts to ensure that all the f_i are identical and equal to a single increasing function f. Normalizing f such that f(0) = 0 and f(1) = 1 then yields the given representation, a function that adds up the v_i's or, equivalently, counts the number of dimensions for which $v_i = 1$.

This reinterpretation of the Pattanaik and Xu result as an additive representation theorem suggests some interesting generalizations. First, if R is an ordering satisfying simple monotonicity and simple independence, might there exist an additive representation $\Sigma_i f_i(v_i)$ where each f_i is increasing and normalized to $f_i(0) = 0$ or, equivalently, an additive representation of the form $\Sigma_i \alpha_i v_i$ where each alternative has a weight or value $\alpha_i > 0$? This would lead to a weighted cardinality rule where instead of counting the number of alternatives in a given set, one would add up the weights associated with its members to indicate the aggregate desirability of the given set.

Alternatively, suppose that we allow v_i to range continuously between 0 and 1, and interpret v_i as the *degree of membership* of the ith element in the given *fuzzy set* associated with the membership vector v (Zadeh, 1965). Let R be an ordering on the domain of fuzzy set vectors $V' = \{v \sum R^n: 0 < v \leq 1\}$. We can directly extend the anonymity, strict monotonicity and the independence axioms to the larger domain of fuzzy sets. It would be interesting to investigate whether the following conjecture is true: R is an (continuous) ordering on V' satisfying anonymity, strict monotonicity and independence if and only if R is represented by $F(v) = \Sigma_I f(v_i)$ for some strictly increasing function f: $[0,1] \rightarrow [0,1]$ with f(0) = 0 and f(1) = 1. If it were true, then interpreting $f(v_i)$ as the "subjective" degree of membership where v_i is the "objective" level, this would provide a justification for a freedom ranking that is represented by a simple sum of subjective degrees of membership across all possible alternatives.

III.3. Reflections on Counting as a Measure of Freedom

The intriguing result of Pattanaik and Xu (1990) has given rise to a secondary literature that critiques this result and suggests alternatives.[12] Perhaps the most natural interpretation of the result (and its general

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11 See for example Blackorby, Primont and Russell (1978).

12 For example Bossert (2000) points out that Pattanaik and Xu presume that there is no uncertainty; that individuals completely control their choices and no undesirable option has the slightest chance of being implemented unintentionally. Klemisch-Ahlert's (1993) spatial approach argues that what should be counted are not the actual number of options, but

approach) is that it is confronting the problem of measuring freedom when the informational basis for doing so is unrealistically thin. More specifically, if individual alternatives are indistinguishable from one another (as embodied in simple anonymity), it shows that the possibilities for constructing useful freedom rankings are rather limited, namely, the cardinal ranking (which identifies greater freedom with a greater number of alternatives), the trivial ranking (which is indifference between all sets), or some censored cardinality rule in between (which counts numbers of alternatives up to a certain level). Given the general acceptability of the monotonicity and semi-independence axioms, there is little ambiguity in this result. The criticism that these measures ignore other aspects important to freedom is valid but has little salience if we truly believe no other aspects are available.

However, the situation is rather different if we take the empirically defensible view that information on the quality of alternatives *is* readily available in the form of the agent's preference ordering. Then the onus would be on supporters of the simple anonymity axiom to try to explain why the agent's preferences over alternatives should be entirely ignored in measuring the freedom of singleton sets. To be sure, all singleton sets have the same cardinality; but since the quality of the elements can be vastly different, maintaining indifference would be tantamount to assuming that quality does not matter at all. In the next few sections we explore alternative approaches that take into account the quality of alternatives as represented by preference relations over alternatives. Interestingly, counting procedures will play a role in the preference-based approaches, but as part of a method of representing preferences and their associated freedom rankings.

IV. Preference and Indirect Utility

The previous section focused on several possibilities for measuring freedom when there is no basis for discerning the quality of different alternatives, and hence they are treated symmetrically. This section begins by assuming that the agent distinguishes between alternatives with the help of a preference ordering. The presumption is that the agent will select a best element from a given opportunity set and hence the comparison of sets entails a comparison of their respective best elements. We alter the Pattanaik and Xu (1990) axioms to obtain a set of properties satisfied by the resulting indirect utility freedom ranking R^U and then note that the ranking is axiomatically characterized by these properties. We explore the relationship between R^U and the cardinality ranking R^C, and construct a series of alternative freedom rankings by combining the two components of freedom rankings – quality and quantity – in various ways.

IV.1. Rankings and Axioms

Let R_1 denote the agent's preference relation over X. How does the existence of a preference ordering impact the resulting freedom ranking? The precise way that R_1 figures into the construction of R will depend crucially on the structure of the selection process that eventually converts a set of alternatives into a specific alternative. For example, suppose that the process is one in which, subsequent to the agent's choice of opportunity set, an alternative is randomly selected by nature and presented to the agent. In this case, the agent cannot predict which outcome will be selected and may wish to choose the menu with the highest expected utility level (for some utility representation of R_1). This would not be much of a freedom ranking since adding an element with below average utility could lower one's ability to achieve a preferred outcome. Alternatively, suppose that while the agent has the power to determine the menu of alternatives, the selection process consists of, say, a mortal enemy picking an alternative from the menu. The agent can certainly predict the outcome that would be selected (the worst) and

rather the characteristics of the options in the set, or the dissimilarity between sets. Pattanaik and Xu (2000) also extend their result beyond the finite sets case.

would likely want to choose a set to maximize the minimum utility. Once again, monotonicity would be forfeited and greater freedom of choice can only hurt the agent – all due to the assumed selection process.

In contrast to the above examples, the selection process assumed here provides an agent with the full measure of information and control in selecting an alternative from the set. When a given opportunity set A is chosen the agent accurately predicts that a best alternative will be selected, and judges the set according to this predicted outcome. The indirect utility freedom ranking R^U compares sets in terms of their best elements: the one with the better best element has greater freedom. Since the underlying preference relation R_1 is an ordering, it follows that the derived freedom ranking R^U is also an ordering. Note that the addition of an alternative will either increase freedom or leave it unchanged, hence the indirect utility ordering satisfies monotonicity (but not strict monotonicity). The addition of one or more new alternatives to any sets A and B satisfying A R^U B will not totally reverse the order of the ranking, and so the indirect utility ordering also satisfies semi-independence (but not independence). On the other hand, so long as R_1 is not completely indifferent across alternatives, simple anonymity will be violated. Instead, the indirect utility ordering satisfies the following natural axiom:

> <u>Extension Axiom</u> For any a, b in X, we have {a} R {b} if and only if a R_1 b.

This axiom requires that in comparisons of singleton sets, the freedom ranking R must echo the judgments of the preference ordering R_1.

There is an additional axiom satisfied by the indirect utility ranking, namely, if a set A has as much freedom as either B or D, then A has as much freedom as the union of B and D. This is due to the fact that if A has an element that is weakly preferred to the best that either B or D has to offer, then its best element will surely weakly dominate the best alternative in B ∪ D. We also define a restricted version of this requirement that restricts A and D to be singleton sets. In symbols, the two axioms are as follows:

> <u>Simple Composition</u> Let A, B, and D be sets in Z such that $|A| = |D| = 1$. If A R B and A R D then A R (B ∪ D).

> <u>Composition</u> Let A, B, and D be sets in Z. If A R B and A R D then A R (B ∪ D).

Notice that this type of axiom is violated by the cardinality ranking, since even if A is considered to have more freedom than B and D individually, the act of combining the two will force the ranking to be reversed if the combined number of elements exceeds the number in A. The exclusive focus on quantity and not quality is what allows this to occur. On the other hand, the indirect utility ranking ignores all but the best alternatives in a given set, and this ensures that A will have greater freedom than B ∪ D.

We have listed four axioms satisfied by R^U: Two that were part of the previous axiomatic framework developed for the cardinal freedom ranking; one that is a natural replacement for simple anonymity when information on the quality of alternatives is provided by a preference relation; and a composition property that makes sense in the present context where the quality of options is paramount. We have the following result.

> <u>Theorem 5</u> R is a quasiordering satisfying simple semi-independence, simple monotonicity, simple composition and extension if and only if R is R^U.

<u>Proof</u> It is clear that R^U satisfies each of the required axioms. Now suppose that R is any quasiordering satisfying simple semi-independence, simple monotonicity, simple composition and extension. Notice that by previous arguments R must satisfy semi-independence and monotonicity. Now consider any A and B in Z and let a be a best element of A under ordering R_1 so that a R_1 a' for all a' ∑ A. By repeated

application of simple composition and extension we have {a} R A. A similar argument for any best element b of B yields {b} R B. By monotonicity it follows that {a} I A and {b} I B. By transitivity, then, A R B if and only if {a} R {b}, or equivalently a R_1 b. It follows, then, that R is simply R^U.

This theorem says that the indirect utility ordering is the only quasiordering that satisfies the standard independence and monotonicity axioms, the preference-based extension axiom, and the intuitive composition axiom. Notice that this result, like the characterizations of R^C above, does not explicitly assume that the freedom ranking is complete. Rather, it derives completeness from the combination of the axioms, none of which individually demands it.

It is instructive to analyze this theorem in the special case where the preference relation R_1 is complete indifference. Under this specification for R_1, the extension axiom is tantamount to simple anonymity, and Theorem 5 would entail all the axioms from Theorem 3. Consequently, the latter result would imply that R would have to be a censored cardinal ranking R^k for some k = 1,…,n. But it is clear that none of the rankings R^k with k > 1 satisfies the simple composition axiom; indeed, we know that {a} R^k {c} and {a} R^k {d} for any distinct a, c, d in X, and yet {c, d} P^k {a} for every k > 1. Hence this additional requirement rules out all possibilities apart from R = R^k with k = 1, or equivalently, R is R^T the trivial freedom ranking. And this is indeed the ranking that Theorem 5 would predict in the special case where the preference relation regards all alternatives as being indifferent to one another.

It is straightforward to show that R^U satisfies the following axiom.

<u>Consistency</u> Let A, B, C and D be elements of Z. Then A R B and C R D implies (A ∪ C) R (B ∪ D).

In other words, if the freedom ranking registers an increase (or stays the same) when B is replaced by A, and D becomes C, then it must also register an increase (or stay the same) when the two higher freedom sets (and the two lower freedom sets) are combined. This requirement is not dissimilar to subgroup consistency requirements used in the measurement of inequality, poverty and welfare.[13] It is easy to show the following result.

<u>Corollary</u> R is a quasiordering satisfying consistency, monotonicity, and extension if and only if R is R^U.

The Pattanaik and Xu characterizations of the cardinal freedom ranking R^C relied on independence, strict monotonicity and anonymity; this result shows that the indirect utility ranking R^U for the posited preference relation R_1 is obtained when strict monotonicity is relaxed to monotonicity, independence is altered to consistency, and anonymity is transformed into extension, which ensures that the ranking over singleton sets follows R_1.

IV.2. Representations of Indirect Utility Freedom

The usual indirect utility function from consumer theory can be interpreted as a measure of freedom that ranks budget sets according to their maximum attainable utility levels. We can follow this approach to obtain a natural counting representation for R^U. For any alternative a in X, let $L_1(a) = \{a' \in X \mid a \, R_1 \, a'\}$ denote the lower contour set of a. Define the utility function u: X → R by u(a) = $|L_1(a)|$, the number of alternatives in the lower contour set. It is a simple matter to show that u represents the preference ordering R_1, i.e., a R_1 b if and only if u(a) ≥ u(b). Indeed, in a world where the only

13 See, for example, Foster and Sen (1997). Note that consistency implies composition.

information that is available about R_1 is ordinal, the utility representation u is an especially intuitive indicator of the desirability or quality of an alternative in terms of the number of alternatives it beats or ties.[14]

One can extend this function to obtain a representation of the ordering R^U on opportunity sets as follows. For any A in Z, define $L_1(A)$ to be the lower contour set of a best alternative in A, i.e., $L_1(A) = L_1(a)$ for some best element a in A. $L_1(A)$ is the "free disposal hull" of A found by taking the union of the lower contour sets of its elements. Extend u to Z by setting $u(A) = |L_1(A)|$, so that the utility of a set is simply the number of alternatives in the free disposal hull of A. We claim that u(A) represents the indirect utility ordering R^U on Z. Indeed, suppose that A R^U B. Then, by the definition of R^U, we know that any best element a of A is weakly preferred to any best element b of B. Hence $L_1(B) \subseteq L_1(A)$ and so $u(B) = |L_1(B)| \leq |L_1(A)| = u(A)$. Alternatively, if it is not the case that A R^U B, then by completeness, any best element of A must be strictly dominated by any best element of B, and so $L_1(A) \subset L_1(B)$ which implies that $u(A) = |L_1(A)| < |L_1(B)| = u(B)$. Consequently A R^U B if and only if $u(A) \geq u(B)$, and so u represents R^U.

This representation of R^U is similar to the one presented above for R^C in that it is based on a simple counting of alternatives. However, instead of ignoring the quality dimension of freedom (as R^C does by using $|A|$) it focuses on the quality or "effective freedom" of a set by including all alternatives that are weakly dominated by its best element, and then measuring freedom as the size of this more expansive set. The result is an indirect utility evaluation of freedom based on the above utility representation of R_1 (which interprets the utility of an option as the number of alternatives it weakly dominates). Note that the difference $u(A) - |A|$ gives the number of alternatives that are "effectively," but not actually, in the set. This margin can be interpreted as the error with which quality is measured by a set's cardinality measure, and its value can range from 0 to $n - |A|$.

IV.3. Combining Cardinality and Indirect Utility

The freedom orderings R^C and R^U often present conflicting judgments due to their different emphases on the quantity and quality dimensions of freedom. For example, an additional alternative always has a positive marginal impact on freedom under R^C, but it increases freedom under R^U only if the alternative has high enough quality (and otherwise has no impact). Singleton sets exhibit the lowest level of freedom under R^C, while they range from the lowest to the highest levels of freedom under R^U. Many authors have suggested that a synthetic approach might be preferable to a single-minded concentration on quantity or quality alone, and we present the two principal candidates below. The first is a traditional *aggregation* exercise, which combines the two freedom measures into a single indicator that balances the effects of the two components when they disagree. The second is a standard *intersection* approach, which removes from consideration comparisons for which the two central rankings disagree and creates an unambiguous ranking containing gaps of incompleteness.

Perhaps the simplest method of obtaining a new freedom ranking from R^C and R^U is to aggregate the functions that represent the two orderings. Let $u(A) = |L_1(A)|$ and $c(A) = |A|$ be the functions representing orderings R^U and R^C, respectively, and let $g{:}R_+^{\,2} \rightarrow R$ be any function that is strictly increasing in each argument. Then the aggregate freedom measure g(u(A),c(A)) and its associate ordering R_g takes both quantity and quality into account when it evaluates menus of alternatives and satisfies strict monotonicity. For example, if $g(u, c) = (u+c)/2$, then freedom measure would be a simple average of the number of alternatives in A and the number of alternatives dominated by the best element of A (under

14 See the related representation of Rader (1963). Note that we are not endowing u with a special normative significance. It is a convenient representation of R_1.

the preference ordering R_1). Note, though that even though both constituent orderings satisfy semi-independence, R_g need not.[15]

While general aggregation approach has not been explicitly studied in the literature, special cases have been investigated. Bossart, Pattanaik and Xu (1993) consider the freedom rankings obtained by applying the two lexicographic orderings on the space of (u,c) pairs, namely

$A\ R_u\ B$ if and only if $u(A) > u(B)$ or $[u(A) = u(B)$ and $c(A) \geq c(B)]$

$A\ R_c\ B$ if and only if $c(A) > c(B)$ or $[c(A) = c(B)$ and $u(A) \geq u(B)]$.

The first of these is the "utility first" lexicographic ordering which primarily relies on the quality indicator to make judgments, but when two sets have the same utility level, it uses the quantity indicator to break the tie. The "cardinality first" ranking proceeds in the opposite order, with the quantity indicator as the primary basis for making comparisons, but with quality being used to break the ties. As an example, suppose X contains three alternatives, x, y, z, with $x\ P_1\ y\ P_1\ z$. Then $\{x\}\ P_u\ \{y,z\}$, illustrating R_u's overriding emphasis on quality, while R_c's contrary focus on quantity yields $\{y,z\}\ P_c\ \{x\}$; and yet $\{x,z\}\ P_u\ \{x\}$ and $\{x\}\ P_c\ \{y\}$ which indicates the sensitivity of each to their respective secondary measures when the levels of the primary measure are equal. It is also not difficult to see that R_u and R_c are examples of orderings R_g generated by an aggregator function g. Indeed, $g_u(u,c) = nu + c$ represents R_u (since the large weight on the u ensures that an increase in u cannot be counteracted by a lower level of c, while if u is unchanged the ranking depends entirely on c); similarly, $g_c(u,c) = nc + u$ represents R_c (by the same reasoning).[16]

At the other end of the spectrum is the intersection approach, which instead of balancing the gains and losses of the two components when they disagree, reaches no decision at all. Bossart, Pattanaik and Xu (1993) applied the vector dominance quasiordering in the space of (u,c) pairs, to obtain the dominance quasiordering R_d is defined by

$A\ R_d\ B$ if and only if $(u(A),c(A)) \geq (u(B),c(B))$

or, equivalently, $A\ R_d\ B$ holds if and only if both $A\ R^U\ B$ and $A\ R^C\ B$ hold. To be sure, conflicts between the two freedom orderings will occur, for example, when one of the sets has just a few elements, but they are high ranking, while the other has many elements, and they are low ranking. The resulting quasiordering is not complete, but since it is more likely for the two components u(A) and c(A) to move weakly in tandem than it is for them to move strictly contrary to one another, it is not disastrously incomplete.

The characteristics of the three freedom rankings R_u, R_c, and R_d, and the interrelationships among them, are well known from analogous discussions of the lexicographic and dominance rankings in two-dimensional space. Obviously, the lexicographic relations are orderings, whose singleton indifference sets in (u,c) space correspond to equal sized opportunity sets whose best elements under R_1 are indifferent under R_1. Note that the quasiordering R_d is a subrelation of both R_u and R_c, and indeed corresponds to their intersection quasiordering (so that $A\ R_d\ B$ holds if and only if $A\ R_u\ B$ and $A\ R_c\ B$). Similarly, R_d is a subrelation of any freedom relation R_g generated by an aggregation function g, and

15 For example, suppose that A has c = 3 and its best alternative has u = 7, while B has c = 4 with u = 5, so that the g(u,c) = (u+c)/2 is larger for A. If an alternative with utility of 8 is added to both, then the order is reversed, violating semi-independence.

16 The lexicographic orderings are representable here due to the discrete nature of u and c.

indeed R_d is also the intersection quasiordering across all such R_g. This holds true even in the extreme case where the preference relation R_1 is completely indifferent between alternatives, so that the indirect utility ordering R^U is the trivial freedom ranking R^T and each of the composite rankings is identical to the cardinality ordering R^C. To rule out this unusual case, and to simplify the subsequent discussion, let us assume that there are distinct alternatives x, y, and z in X such that x R_1 y and y P_1 z.

It is easy to show that all three of these composite rankings satisfy the same subset of the previously defined properties, namely, extension, semi-independence, monotonicity and strict monotonicity. They all violate anonymity and its simple version (by Theorem 3), composition and its simple version (by Theorem 5), consistency (by Theorem 6), and independence and its simple version, which can be shown directly by example. They inherit semi-independence and monotonicity from both component orderings R^U and R^C. Unlike R^U, which only satisfies monotonicity, each composite ranking satisfies strict monotonicity through its sensitivity to R^C. Unlike R^C, which satisfies simple anonymity, each composite ranking satisfies the extension axiom through its sensitivity to R^U. Bossart, Pattanaik and Xu (1993) find additional axioms that allow the three composite rankings to be uniquely characterized from among all freedom rankings.

IV.4. Other Views of Quality

The last section illustrated how quality (as represented by the utility of the best element) can be combined with quantity (the number of elements in a set) to create freedom rankings with various desirable characteristics. Other authors drop quantity as an explicit dimension of freedom and consider alternative notions of quality that take into account the entire distribution of utilities from elements in the set, not just the maximum utility. We now present the various proposals that proceed along this line.

As discussed above, we can represent every set A by a vector v of 0's and 1's, where $v_i = 1$ indicates that the corresponding alternative x_i is in the set A, while $v_i = 0$ indicates that x_i is not. Let q be the corresponding utility vector defined by $q_i = v_i u(x_i) = v_i|L_1(x_i)|$ for i = 1,…,n, so that q_i is the utility of x_i in set A (or 0 if it is not). Reordering the entries of q from highest to lowest, we obtain the *ordered version* of q, formally defined as the vector $\hat{q}$ satisfying $\hat{q}_1 \geq \hat{q}_2 \geq \ldots \geq \hat{q}_n$ and $\hat{q} = \Pi q$ for some nxn permutation matrix Π. We will call $\hat{q}$ the *(ordered) quality vector*, since $\hat{q}_i > 0$ is the quality of the ith best alternative in A, while $\hat{q}_i = 0$ indicates that A has no ith best element.[17]

Note that the first entry $\hat{q}_1 = \max_i \hat{q}_i = u(A)$ is the quality or utility of the best element of A, which is our measure of indirect utility freedom. The approaches discussed in this section extend consideration to the entire quality vector $\hat{q}$ in evaluating the freedom of A. Once again there will be two main candidates for combining the n dimensions of $\hat{q}$ to obtain a freedom ranking: *aggregation* and *intersection*.

One natural approach to aggregation would be to use a strictly increasing function h: N → R to combine the dimensions of $\hat{q}$ into an overall measure of freedom F(A) = h($\hat{q}$). For example, the measure F(A) obtained when $h(\hat{q}) = \Sigma_i \hat{q}_i$, or more generally $h(\hat{q}) = \Sigma_i w_i \hat{q}_i$ for some weights w_i that are weakly decreasing in i to capture the intuition that the first best should have at least as much weight as the second best, etc. Now the resulting ranking clearly satisfies monotonicity; however, as before, there may be problems with semi-independence. On the other hand, if the weights w_i decrease abruptly enough in i, then the resulting measure would represent the *leximax*, a semi-independent ordering R_m discussed by

[17] There may ambiguity in defining a unique ith best element when R_1 is indifferent between two elements of A; however, since the respective quality levels would then be the same, we may arbitrarily select one of the two without loss of generality.

Bossart, Pattanaik and Xu (1993).[18] This ranking first compares the quality of the best elements from each set and, if equal, compares the quality of the next best elements and so on until a lexicographic comparison of quality vectors is obtained. Where L is the standard "earlier is better" lexicographical ordering, R_m can be defined by

$$A \; R_m \; B \text{ if and only if } \hat{q} \; L \; \hat{q}'$$

given that $\hat{q}$ is the quality vector of A and $\hat{q}'$ is the quality vector of B. Note that A P^U B implies A P_m B, in which case both rankings will ignore the quality and quantity of lower quality alternatives. But when A I^U B we could have either A P_m B or B P_m A depending on whether the inequality is $\hat{q}_i > \hat{q}_i'$ or $\hat{q}_i < \hat{q}_i'$ the first time that $\hat{q}_i$ departs from $\hat{q}_i'$. Again, once a strict inequality is obtained one way or the other, the remaining elements have no impact.

Sen (1991) applies the intersection approach to obtain a freedom ranking R_s that is equivalent to comparing sets A and B using vector dominance over their respective ordered quality vectors, that is,

$$A \; R_s \; B \text{ if and only if } \hat{q} \geq \hat{q}'.$$

There are three cases (and hence interpretations) of the constituent inequality $\hat{q}_i \geq \hat{q}'_i$: (i) If $\hat{q}'_i > 0$, then $\hat{q}_i \geq \hat{q}'_i$ indicates that ith best member of A is as good as the ith best member of B; (ii) If $\hat{q}_i > 0 = \hat{q}'_i$, then A contains an ith best element, but B does not; and (iii) if $\hat{q}_i = 0 = \hat{q}'_i$, then neither A nor B has an ith best element. Sen's (1991) original definition has A R_s B holding whenever there exists a one-to-one correspondence from B to A, with each element of B being weakly dominated by its corresponding element from A (according to R_i). However, it is clear that this is equivalent to the definition via quality vectors, since (i) above implicitly defines a correspondence meeting Sen's requirements.

Through the first entry of the quality vector, it is easy to see that A R_s B implies A R^U B for any opportunity sets A and B. The remaining n-1 requirements act to ensure that the original ranking over best elements is robust to the other dimensional comparisons. If there is some ith order comparison that strictly disagrees with the original judgment, then the two sets are declared to be noncomparable. On the other hand, A and B can be indifferent according to R^U, and yet A P_s B through a strict inequality for some higher order comparison.[19] It is also clear that A R_s B implies that A R^C B, and hence A R_d B, where R_d is the quality/quantity dominance ranking defined above. It is interesting to note that since the quality vectors are ordered, the degree of incompleteness of quasiordering R_s is quite low, and certainly not much very worse than R_d; indeed, for the case of five strictly ordered elements, R_d has approximately 10% of set comparisons being noncomparable, while R_s has about 15%.

To discuss the properties satisfied by R_m, and R_s, let us once again assume that there are three alternatives x, y, and z, such that x R_1 y P_1 z. Both freedom rankings satisfy the fundamental properties of monotonicity and semi-independence as well as strict monotonicity and the extension axiom. At the same time, we know from the above theorems that both rankings violate anonymity and its simple version, composition and its simple version, and consistency. Curiously, both also satisfy independence. Bossart, Pattanaik and Xu (1993) find additional axioms that uniquely characterize the leximax ordering from among all freedom rankings.

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18 One such representation of R_m would be $h(\hat{q}) = n^{n-1}\hat{q}_1 + n^{n-2}\hat{q}_2 + \ldots + n\hat{q}_{n-1} + \hat{q}_n$.

19 This uses the standard definition of P_s as the asymmetric part of R_s. Sen (2002) attempts to avoid this possibility by defining a strict freedom ranking that strictly dominates in all dimensions, including the first.

IV.5. Reflections on Indirect Utility as a Measure of Freedom

An economist confronting the question of how to compare sets of alternatives in this environment would naturally begin with the indirect utility measure. This approach is very clear about priorities: the utility of the best alternative is the basis of comparison and nothing else matters. The quality or quantity of other alternatives that happen to be in or outside the set are not relevant, and there is no willingness to sacrifice the indirect utility to obtain additional or higher quality, unused alternatives. A set is equivalent to the singleton set containing its best element. The motivating economic model – of an agent with well-defined preferences who is evaluating various sets in anticipation of selecting an alternative from it – drives the evaluation inexorably in this direction. At the end of the day, the agent can only select a single alternative, and it is intuitive to identify the value of this single alternative with the value of the set to the agent.

Despite the compelling plausibility of the underlying model, it is not entirely clear why this instrumental value should be the *only* basis for evaluating sets in terms of the freedom they provide. As noted above, there is wide scope for incorporating all the available information, including preference-based quality and quantity data, to construct measures of freedom that go beyond indirect utility. The axiomatic descriptions of these approaches may go some distance in uncovering the unique features of each measure and perhaps in selecting among them. However, without a conceptual framework for understanding and justifying the departures from indirect utility, the entire exercise appears to be ad hoc. The key question is: How do the unchosen alternatives in a set provide value to the agent (either intrinsically or instrumentally), even though, with the given preference relation, they remain unselected? It is to this question that we turn in the next section.

V. Plural Preferences and Effective Freedom

The cardinality of an opportunity set is a natural way of measuring freedom in the absence of information about the agent's preferences. Yet once a preference relation has been specified, and the agent can freely select the most preferred alternative from the set, it is not obvious why the number of unchosen alternatives, or even their quality, should matter. The indirect utility measure of freedom takes the view that an opportunity set has the same level of freedom as a singleton set containing its best element, and since the additional alternatives would never be selected by the agent, they do not add to the agent's opportunity freedom. However, this may no longer be justified if the agent were to have a *collection* of potential preference orderings and there were uncertainty over which of these preferences would arise. In the resulting *plural preference* environment, a best element can depend on the realized preference ordering; an unchosen alternative may have value in providing insurance or flexibility before the uncertainty is fully resolved. This approach is motivated by the literature on *preference for flexibility* as presented by Kreps (1979) and Koopmans (1964); it has close conceptual links to the key notion of option value in economics and finance (Weisbrod 1964; Malkiel and Quandt 1969).[20] There are many ways of incorporating the new information on preferences into a freedom ranking. This section presents several plural preference methods that are based on the *intersection* or *aggregation* of preferences.

We begin with the *effective freedom* approach of Foster (1993), which constructs a freedom ranking by taking the intersection of the indirect utility rankings arising from a set of preferences. Under this approach, one set will have greater freedom than another if, no matter the preference relation, it has at least as high indirect utility and, for some preference relation, it is higher. At one extreme (where there is a single preference orderings), this approach yields an indirect utility ordering; at the other (where all

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[20] See also Sen (1980/81).

preferences are possible) it tends towards the set inclusion quasiordering. The properties of the freedom ranking are explored, and several natural representations are given. One partial representation based on a counting of alternatives in "effective opportunity sets" extends the effective freedom ranking to a complete ordering that is the indirect utility ordering at one extreme and the Pattanaik and Xu (1990) cardinal ordering at the other.

The alternative intersection approach by Sen (2002) is then discussed, which first applies the intersection approach to the collection of preference orderings and then extends the resulting quasiordering over alternatives to a quasiordering over opportunity sets. It is noted that the resulting freedom ranking is generally more incomplete than the effective freedom ranking, but when it does hold, much stronger statements about the freedom levels of the two sets can be made. Note that neither of these approaches requires the agent to make an assessment of the likelihood of the various possible preferences. Arrow (1995) follows more closely the lead of Kreps (1979) in assuming that each preference relation (as represented by a utility function) has a given probability of arising, and hence the indirect expected utility may be used in evaluating opportunity sets. Under this aggregation approach, one set has greater freedom if the expected utility level of its best element is above the level obtained from the second set. We note that Arrow's approach reduces to an indirect utility freedom ordering when a particular utility function has probability 1; and it can generate the cardinality freedom ordering in the special case when all preferences are possible and equally likely.

V.1. Rankings and Axioms

The effective freedom approach of Foster (1993) begins with a collection $\mathfrak{R} = \{R_1,\ldots,R_m\}$ of complete orderings over X, where the number m of preference relations can vary from 1 to all logically possible preferences orderings. $\mathfrak{R}$ is interpreted as the set of all *potential* preferences that the agent may have, although alternative interpretations are possible and are consistent with the formal implications of the approach.[21] The *actual* preference relation will be revealed to the agent before it is time to select an alternative from the chosen opportunity set and, indeed, from this point on the resulting selection process will be entirely analogous to indirect utility approach: the agent selects a best element from the opportunity set according to the preference relation. However, at the time that the opportunity set is evaluated, the agent only knows the collection $\mathfrak{R}$ of possible preferences and *not* the one that eventually obtains.

Let us return to the occupational choice problem facing the recent graduate Jill. If Jill begins with only a singleton choice set, when does adding an alternative strictly improve her freedom and when does it not? Clearly, if the original set contained a single alternative (say civil engineering) which under all of her possible preferences in $\mathfrak{R}$ were considered to be at least as good as the new alternative (say teaching), then the opportunity set with both alternatives would offer no additional effective freedom than the single choice of civil engineering alone. Under all potential realizations of preferences, Jill will do as well with the singleton set as with the two-vocation set. Alternatively, suppose that Jill has at least one preference ordering in her set that ranks teaching strictly above civil engineering. Then adding the additional alternative of teaching to the civil engineering singleton set would certainly expand Jill's opportunity freedom since she would be free to select a strictly preferable vocation if that preference ordering happened to emerge.[22]

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21 Other interpretations are possible. See for example Pattanaik and Xu (1998).

22 Note that this discussion is entirely dependent on the assumption that the process aspect of freedom has not been compromised so that Jill can freely select from the opportunity set once her preferences have been realized. Also, we are not restricting the mechanism by which preferences are realized, apart from the requirements that all preferences in $\mathfrak{R}$ are

For any given preference ordering R_j in $\mathfrak{R}$, let R_j^U denote the associated indirect utility ranking on Z. One opportunity set A is said to have as much *effective freedom* as a second opportunity set B, written A R* B, if A has as much indirect utility freedom as B for all allowable preferences; i.e.,

$$A\ R^*\ B \text{ if and only if } A\ R_j^U\ B \text{ for all } R_j\ \Sigma\ \mathfrak{R}.$$

This means that no matter which preference ordering R_j in $\mathfrak{R}$ happens to arise, the opportunity set A has an element that is at least as good as all the elements of B according to R_j (although this option may vary with R_j). Note that since R* is the intersection of a collection of complete orderings over opportunity sets (namely, the indirect utility rankings), it must be a quasiordering (i.e., transitive and reflexive, but not necessarily complete). As is typical of quasiorderings, the symmetric part I* requires I_j^U to hold for all constituent preference orderings, while the asymmetric part P* requires R* in conjunction with P_j^U for some preference ordering in $\mathfrak{R}$.

It is easy to see that R* will generally violate the Pattanaik and Xu (1990) axioms: Independence, since removing an unambiguously best option from both sets may convert I* to P*; strict monotonicity, since adding a dominated option leads to I* and not P*; and anonymity, since P* or noncomparability across singleton sets can easily arise. On the other hand, R* inherits semi independence, monotonicity, composition and consistency from its constituent indirect utility orderings. The extension property is no longer applicable since there are now many potential preference relations, and no one of them determines the freedom ranking over singleton sets. However, the property can be generalized to apply to the plural preference environment as follows:

<u>Plural Extension Axiom</u> For any a, b in X, we have {a} R {b} if and only if a R_j b for all $R_j\ \Sigma\ \mathfrak{R}$.

This axiom posits that if all preference orderings in $\mathfrak{R}$ agree that one alternative is at least as good as another, then the freedom ranking over the associated singleton sets must be consistent with this consensus opinion. Moreover, in the case where two preference orderings disagree, the associated singleton sets are unable to be compared using the freedom ranking.

A second property concerns the role of sets of options that are noncomparable under R* in creating sets that have strictly greater freedom. As in the example where Jill is not sure how to rank teaching and civil engineering, let us consider opportunity sets A and B that cannot be compared using R*. By monotonicity we know that (A ∪ B) R* B and hence (A ∪ B) R_j^U B for all $R_j\ \Sigma\ \mathfrak{R}$. Since A and B are not comparable this implies that B $P_{j'}^U$ A for some $R_{j'}\ \Sigma\ \mathfrak{R}$. But by the transitivity of ordering $R_{j'}^U$ we have (A ∪ B) $P_{j'}^U$ A, and hence (A ∪ B) P* A by the definition of P*. It follows that R* satisfies the following property:

<u>Semi Strict Monotonicity</u> Let A and B be elements of Z. If A ⊂ B then B R A; and in addition if A and B\A ≠ ∅ are noncomparable under R, then B P A.

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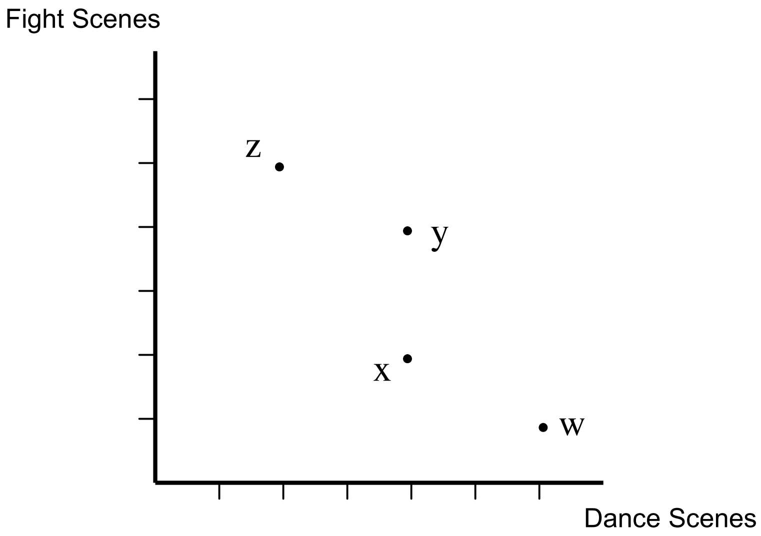
considered to be relevant at the time an opportunity set is being evaluated; and only one preference ordering will be relevant when an alternative is selected from the set. Nature could select the preference ordering, or the person's mother, or the person. Which of these is correct may well impact our evaluation of process freedom of the final choice; but our procedure is designed to measure only the opportunity freedom across sets, and abstracts from the way the resolution of preference uncertainty can directly impact process freedom.

In addition to monotonicity, this axiom requires that whenever an opportunity set A is merged with a second, noncomparable opportunity set B\A, we obtain a new set B having strictly greater freedom. So if Jill is unable to rank the singleton options of teaching and civil engineering, having both options is strictly freedom enhancing. It would be interesting to explore whether these latter two axioms could characterize R* among all freedom quasiorderings satisfying consistency.

The effective freedom ranking R* is generally incomplete, and its ability to make comparisons depends on the extent to which the preferences in $\mathfrak{R}$ (or more precisely the indirect utility orderings) agree. If $\mathfrak{R}$ has only a single preference ordering, then R* will be an indirect utility ranking and hence complete. If $\mathfrak{R}$ contains all logically possible preference orderings on X, then R* becomes set inclusion ranking, so that $A \supseteq B$ if and only if A R* B. This is the most incomplete quasiordering consistent with monotonicity. An intermediate, six alternative example based on Sen (1990) shows how agreement across preference orderings expands the reach of R* beyond set inclusion. Suppose C = {g,t,w} contains "great," "terrific," and "wonderful" alternatives while D = {b,a,d} contains "bad," "awful," and "dismal" alternatives wherein each option in C is regarded by every $R_i \sum \mathfrak{R}$ as strictly preferable to each option in D. Preferences in $\mathfrak{R}$ are otherwise unconstrained. Then the associated effective freedom ranking goes beyond set inclusion in two ways: (i) if A contains an alternative from C while B does not, then A P* B; (ii) if both contain alternatives from C, then the ranking depends only on these alternatives, with A R* B following from $(A \cap C) \supset (B \cap C)$. Of course, C P* D as expected.

A second example based on Foster (1993) shows that the effective freedom ranking R* can compare opportunity sets even when the underlying preferences in $\mathfrak{R}$ are incompatible. Kamala is a Bollywood aficionado who judges movies by the number of dance scenes and number of fight scenes, so that for present purposes a movie is represented as a vector of the two values (so that x = (3, 5) has three dance and five fight scenes). On some nights, Kamala has a strong hankering for choreography and would rather avoid the punch-ups: her "dance-loving" ordering R_1 is represented by $u_1(x) = x_1 - x_2$. On others, she looks forward to rampant fisticuffs but could do without the Bhangra: her "fight-loving" ordering R_2 is represented by $u_1(x) = x_2 - x_1$. The set X = {w,x,y,z} of all possible movies is depicted below. Notice that her two preferences never agree: R_1 ranks the elements as $wP_1xP_1yP_1z$, while R_2 ranks them oppositely as $zP_2yP_2xP_2w$. The intersection quasiordering generated by R_1 and R_2 on X is empty – there is no hope for consensus. In contrast the effective freedom ranking R* on Z, which is the intersection of the indirect utility rankings R_1^U and R_2^U, can make many nontrivial comparisons. For example, consider {w,z} and {x,y}. Clearly {w,z} offers dance-loving Kamala her best element (namely w) and also offers fight-loving Kamala her best alternative (namely z). Consequently, both indirect utility rankings R_1^U and R_2^U agree on the superiority of {w,z} over {x,y}, so {w,z} P* {x,y}. Moreover, since singleton sets are noncomparable, any two-alternative opportunity set has greater effective freedom than either of its singleton components (as required by strict semi monotonicity). We will return to this example when we discuss Sen (2002).

Figure 1: Choosing Movies



V.2. Representations and Measures

The effective freedom ranking R* has several functional representations, each of which helps in interpreting the ranking and linking it to other approaches. The first is a natural representation in terms of utility vectors. Recall the discussion of indirect utility freedom in which the set $L_1(A)$ (containing all choices weakly dominated by some element of A) was used in constructing the utility representation $u(A) = |L_1(A)|$ of R^U. The completeness of R_1 ensures that the effective opportunity sets $L_1(A)$ are nested and that a count of alternatives fully reflects the indirect utility freedom ranking R^U. In the plural preference case we can construct the analogous utility representation $u_j(A) = |L_j(A)|$ for each R_j^U and then define the vector-valued utility function $u(A) = (u_1(A),\ldots,u_m(A))$. It is clear from the definition of R* that

$$A\ R^*\ B \text{ if and only if } u(A) \geq u(B)$$

and hence u is a vector valued representation of R*. The function u maps from Z to the space of utility vectors ordered by vector dominance and is entirely analogous to how Pareto dominance is viewed in utility space. As noted below, this representation has particular relevance to Arrow (1995) in which utility levels are combined to obtain a complete ordering of Z, much as a social welfare function extends Pareto dominance.

The second approach maps from opportunity sets in Z to other *effective opportunity sets* in Z that are then compared using set inclusion. Returning to the indirect utility setting, it is easily shown that for any two opportunity sets A and B, the ranking A R^U B holds exactly when $L_1(A) \supseteq L_1(B)$. In words, when assessing the indirect freedom levels of two opportunity sets, we can simply check whether the associated effective opportunity sets can be ranked by set inclusion. In this way, L_1 can be viewed as a representation of R^U. It turns out that an analogous representation holds for R* in the plural preference setting. For any opportunity set A, define the *effective opportunity set* $D(A) = \cup_{AR^*B}B$ to be the set containing all elements that are found in any set B weakly dominated by A according to R*.[23] D(A) is

[23] Note that A R* D(A) by composition.

equivalently seen as the set of all y ∑ X having the property that no matter which preference ordering R_j ∑ $\mathfrak{R}$ is selected, there is some x ∑ A for which x R_j y. It can be shown that A I* D(A), and that A R* B holds exactly when D(A) $\supseteq$ D(B). In other words, A has the same level of effective freedom as its associated effective opportunity set D(A), and furthermore R* is represented by D where rankings across sets in its range are determined by set inclusion. Note that unlike the indirect utility case (which yields a complete freedom ranking R^U), it is possible for two sets A and B to be unranked by R*, which corresponds to the case where neither D(A) nor D(B) contains the other. This general form of representation will be used below in our discussion of the related approach of Sen (2002).

A third representation uses the enumeration methodology of Section III.2 by which an opportunity set A is described as a vector v(A) of 0's and 1's (where 1 indicates an alternative's presence while 0 indicates its absence). Let $v_D(A) = v(D(A))$ denote the vector corresponding to A's effective opportunity set D(A), and similarly define $v_D(B)$. Then it is immediate that

$$A\ R^*\ B \text{ if and only if } v_D(A) \geq v_D(B)$$

so that v_D is a vector valued representation of the effective freedom ranking. If $\mathfrak{R}$ includes all logically possible orderings, then D(A) = A so that v itself is a vector-valued representation of the effective freedom ranking R* (which in this case is set inclusion). In the special case where $\mathfrak{R}$ has a single ordering, every pair of vectors $v_D(A)$ and $v_D(B)$ can be ranked by vector dominance and R* (or the indirect utility freedom ranking) is complete. This representation is the vector-valued version of the previous representation based on effective opportunity sets.

Define the *counting representation* $u^\#(A) = |D(A)|$ to be the cardinality of the effective opportunity set D(A) or, equivalently, the sum of the entries of the vector $v_D(A)$. This is an intuitive numerical measure of freedom that uses information on preferences from $\mathfrak{R}$ to assess the quality and range of choices in A. It does so by counting the number of elements in X that are dominated by A, and hence that the number that A effectively includes (given $\mathfrak{R}$). It is clear that $u^\#$ *partially* represents R* in the sense that

(i) A I* B implies $u^\#(A) = u^\#(B)$, and

(ii) A P* B implies $u^\#(A) > u^\#(B)$.

This means that the measure $u^\#$ follows the quasiordering R* when it applies, and hence is never inconsistent with its rankings.[24]

The counting representation $u^\#$ induces a complete ordering, say $R^\#$, that extends R* in a natural way and is of some independent interest. If A and B are comparable by R*, then D(A) and D(B) are ranked by set inclusion, and hence the counting representation ensures that $R^\#$ ranks A and B in the same way; if A and B are not comparable under R*, so that neither D(A) nor D(B) lies within the other, $R^\#$ makes the comparison by counting alternatives in the respective effective opportunity sets and ranking accordingly.[25] $R^\#$ thus bases its rankings on information about the quality of options as interpreted by

24 See Aumann (1962, p. 450) who uses (i) and (ii), and Majumdar and Sen (1976) who provide a general discussion of the representation of incomplete rankings. Note that a full representation (with "if and only if") of R* is out of the question $u^\#$ can compare any two sets; this is why the implications are only required in one direction. Of course a partial representation is less informative than a full one, since there are typically many quasiorderings consistent with a complete ordering.

25 This extension is not unlike the way that the Gini coefficient inequality measure extends the Lorenz ranking.

the preferences in $\mathfrak{R}$ and uses counting methods when preference information is not available to make the comparison.

It is instructive to examine the freedom rankings R^* and $R^\#$ in two extreme cases: first where $\mathfrak{R}$ has a single element, so that the decision maker's preference ordering is known with certainty; and second where $\mathfrak{R}$ contains all logically possible preferences, which may be interpreted as having no information on the decision maker's preferences. In the first case, the effective freedom approach is identical to the indirect utility approach, generating the complete ranking R^U that is based on the given preference ordering. And since R^* is complete, there is no scope for extending the quasiordering by counting and hence $R^U = R^* = R^\#$. The counting measure $u^\#$ represents each of the three freedom orderings; it aggregates all the available information on preferences and nothing else. In the second case, the lack of effective information on preferences reduces each effective opportunity set D(A) to A itself. Consequently, R^* is reduced to set inclusion, a minimal freedom quasiordering, while the counting ordering becomes R^C, the cardinality ordering presented by Pattanaik and Xu (1990). With so little information to go on, the effective freedom ranking R^* is squeezed down to the most incomplete freedom quasiordering; the counting measure $u^\#$, then, must perform much of the heavy lifting in extending R^* to $R^\# = R^C$.

Pattanaik and Xu's (1990) cardinality approach to freedom and the traditional indirect utility approach to freedom thus arise as two polar cases of the same method where freedom is measured as the number of options in the effective opportunity set. Pattanaik and Xu are located at one extreme where preference information is essentially absent, while indirect utility is at the other where preference information is exact. The effective freedom approach and its associated counting measure based on the size of effective opportunity sets can go well beyond these two informational extremes, since all intermediate degrees of information in $\mathfrak{R}$ can be accommodated and incorporated into this framework for evaluating freedom.

V.3. Sen's Subrelation and Arrow's Extension

The effective freedom relation is one natural way of evaluating opportunity freedom in the plural preference model. In his Arrow Lectures, Sen (2002) presents an alternative route based on a more stringent criterion for comparing opportunity sets. The result is a subrelation of R^* that is typically less complete and satisfies fewer of the properties discussed above, but conveys new information about the sets it ranks. This information can become especially pertinent when the order of events in the model is altered. Arrow (1995) presents a second approach that extends R^* to a complete ordering of opportunity sets in a plural preference model that has some additional structure. He assumes that the realized preference relation of the decision maker will be drawn according to a known probability distribution; and he assumes that appropriate utility representations have been specified for the preferences. The opportunity freedom of a given set is then measured as the expected indirect utility of an opportunity set.

This section examines the approaches of Sen and Arrow. In particular, we provide alternative interpretations of the rankings, determine the properties they satisfy, and discuss their representations. We use examples to illustrate how the Sen subrelation makes fewer comparisons and Arrow's extension makes more, and then explain and evaluate these departures from R^*.

Sen (2002) presents a freedom relation that ranks opportunity sets whenever an element can be found in one set that is weakly preferred to every element from the second set, irrespective of the preference ordering. In symbols, the Sen freedom ranking R^S is defined by

$$A \; R^S \; B \text{ if and only if there exist } a \sum A \text{ such that } a \; R_j \; b \text{ for all } R_j \sum \mathfrak{R} \text{ and all } \quad b \sum B,$$

By contrast, the effective freedom ranking can be defined as

$$A\ R^*\ B \text{ if and only if for each } R_j \sum \mathfrak{R} \text{ there exist } a_j \sum A \text{ such that } a_j\ R_j\ b \text{ for all } b \sum B,$$

and hence R^S is a subrelation of R^* (i.e, $A\ R^S\ B$ implies $A\ R^*\ B$). Under R^* the dominating alternatives a_i can vary with the preference relation R_j, while R^S requires all the a_j to be the same – an additional restriction that might be called Sen's *uniformity requirement*.[26] Sen (2002) justifies R^S within a model that has an altered timeline for uncertainty resolution, and this argument will be reviewed below. We begin with a discussion of the coverage, representations and properties of the resulting freedom relation R^S.

To see how the uniformity requirement affects the coverage of the freedom relation, let us revisit our previous examples. In Sen's classic example with sets C = {g,t,w} and D = {b,a,d}, the ranking $C\ P^S\ D$ follows immediately from the definition of R^S, since any element of C dominates any element of D for all possible preferences. However, R^S cannot compare C to *itself*, nor even to the menu $\{g,t\} \subset C$, since by assumption no one element dominates g and t for all preferences. This, of course, means that that R^S is neither reflexive nor monotonic, in contrast to R^* and the other relations we have discussed. In the second example, where the effective freedom relation concludes $\{w,z\}\ P^*\ \{x,y\}$, the Sen relation is powerless to compare {w,z} and {x,y}: no *one* alternative in {w,z} is better than each element of {x,y} for both preference orderings. Clearly, R^S is more incomplete than R^* in this key example as well.

Since R^S is a subrelation of R^*, it follows that each representation for R^* constitutes a *partial* representation for R^S. In general, though, since R^S is not a quasiordering, it is impossible to find a *full* representation for R^S analogous to the ones found for the other freedom rankings. It *is* possible to restate the definition of R^S in terms of R^* and its effective opportunity set representation. Indeed, $A\ R^S\ B$ holds if and only if there exists $a \sum A$ such that $\{a\}\ R^*\ B$ and, consequently,

$$A\ R^S\ B \text{ if and only if } D(\{a\}) \supseteq D(B) \text{ for some } a \sum A.$$

Testing for $A\ R^*\ B$ requires a single comparison of D(B) and D(A), while $A\ R^S\ B$ entails a series of comparisons between D(B) and *every* D({a}) for $a \sum A$.

As for the properties of this relation, it is immediate that R^S satisfies transitivity, simple monotonicity, simple composition, and plural extension. It is likewise clear that R^S follows R^* in generally violating the Pattanaik and Xu (1990) axioms of simple anonymity, simple independence, and simple strict monotonicity (and their stronger versions). However, apart from simple composition, it can be verified that the other properties that were shown to be satisfied by R^* are not generally satisfied by R^S.[27] Using R^S instead of R^* can mean foregoing these intuitive properties.

Sen (2002) has emphasized that the appropriateness of a given freedom ranking can depend upon the "sequence of events" in the underlying model. Recall that in the original effective freedom model, the preference uncertainty is resolved *after* the choice of opportunity sets but *before* the selection of a specific alternative from the set. Given this "sequencing," the effective freedom ranking R^* is the natural ranking

26 To be precise, it requires that among all such dominating collections of a_i there must be at least one set with a_j constant across all j. Note that there is no presumption that the resulting uniform a is in any sense a best choice from set A: a could be strictly worse than some a_j' in A according to R_j, or even dominated by some a' uniformly across *all* preferences.

27 For example, to show semi-independence can be violated, consider X = {a,b,c} and $\mathfrak{R} = \{R_1, R_2\}$ with $a\ P_1\ b\ P_1\ c$ and $c\ P_2\ a\ P_2\ b$. Then $\{a\}\ R^S\ \{b\}$ and yet we do not have $\{a,c\}\ R^S\ \{b,c\}$. To see that composition can be violated, let X = {a,a',b,d} and suppose $\mathfrak{R} = \{R_1, R_2\}$ has $a\ P_1\ b\ P_1\ a'\ P_1\ d$ and $a'\ P_2\ d\ P_2\ a\ P_2\ b$. Then $\{a,a'\}\ R^S\ \{b\}$ and $\{a,a'\}\ R^S\ \{d\}$, and yet we do not have $\{a,a'\}\ R^S\ \{b,d\}$.

to use.[28] A different ranking may arise, though, if resolution of the uncertainty is pushed up or back. For example, if the resolution occurs *before* either choice has to be made, this yields the special case of a single preference ordering and suggests the use of the indirect utility freedom relation R^U. Alternatively, suppose that the resolution takes place *after* both choices, so that one would have to select a menu *and* an alternative without knowing which of the preferences would actual obtain. Sen suggests that R^S would be a natural ranking in this alternative plural preference model.

In principle, even with the new order of events we could still rank sets according to R*; however the original justification for that ranking (that a preference-contingent set of choices from A can be found which dominates any such preference-contingent set of choices from B) would have no force. Preference-contingent choices are no longer feasible and their relative desirability has no relevance. Under the new scenario, a single choice must be selected from each set without knowing which preference ordering will eventually obtain, and the single choice can have a different value under each preference ordering. This is easiest to see if we use utility functions, since then every element in X can be evaluated using its vector of associated utilities (one for each preference ordering). Sen's uniformity requirement ensures that there is some alternative a in A having a utility vector that dominates the utility vector of any element in B. If the ultimate choice from A (which may well be a different option entirely than a) delivers a utility vector that is at least as high as a's utility vector, then A can be viewed as giving greater opportunity freedom than B, and this is exactly what R^S conveys.

This justification for R^S may be somewhat less compelling than the argument on behalf of R*, since the method of choice underlying R^S has not been fully specified (while it was for R*). Indeed, the construction of meaningful choice mechanisms and rankings over sets can be a nontrivial exercise in the present context with increased uncertainty. Other considerations may well enter into the way that sets are compared. Sen (2002, p. 614) provides a useful example that incorporates a "preference for safety" choice mechanism. Return once again to the Bollywood example, and consider the vectors of utility pairs that the three elements w, y, and z generate:

	u_1	u_2	min
w	5	-5	-5
y	0	0	0
z	-3	3	-3

In the third column we have listed the minimum utility across the option's two utility levels. Now let us compare the opportunity sets {w,y,z} and {w,z} with the help of the well known "max-min" choice mechanism.[29] From the table we see that the option in {w,z} with the highest minimum utility level is z with -3, while the max-min choice from {w,y,z} is y with 0. Using the same criterion, the choice between these two options would be y, and hence we could argue that {w,y,z}, which contains the safe alternative

28 See Sen (2002, p. 677). To be sure, subrelation R^S might *also* be applied in this environment to distinguish between certain R* comparisons; however, it may not make sense to rule out a given R* comparison just because the uniformity requirement fails. This suggests the use of R* to compare opportunity sets, with the optional use of R^S to discern between different cases when R* applies.

29 See for example Gilboa (2009).

y, is strictly superior to {w,z}, which does not. This illustrates the possibilities if one can identify a very specific choice mechanism; however, the resulting ranking will be entirely contingent on that specification. In contrast, R^S cannot compare {w,y,z} and {w,z} at all, since all utility vectors are noncomparable. Its uniform dominance approach to evaluating freedom simply cannot apply. And if we return order of events to the usual effective freedom setting (with choice of alternative placed after the resolution of uncertainty), then R* will apply, and the two sets {w,y,z} and {w,z} will be seen as providing *equal* effective opportunity. Under this sequencing, y will never be chosen, no matter which preference arises; its "safety value" utility vector of (0,0) is clearly dominated by the utility vector (5,3) of the preference-contingent choice of w under u_1 and z under u_2, which is available from both sets.

Sen's (2002) treatment of plural preferences restricts R* by requiring an additional uniformity condition before rendering a judgment, and generates a subrelation R^S of R*. Many of the characteristic properties of R* are lost in the process, and the resulting relation is less complete. However, R^S provides additional information when it applies, and it is a natural way of ranking opportunity sets in a decision model where the timeline has been altered so that all decisions must be made before the uncertainty is resolved.

Arrow (1995) argues that "the notion of freedom must be grounded in a multiplicity of preferences" but contends that "if the concept of freedom is to have any operational meaning, it must lead to a complete ordering." Drawing upon a stream of literature in economics reaching back to Hart (1942), he provides a simple framework for evaluating opportunity freedom as *flexibility*. This view mirrors the framework of Kreps (1989) in relying upon two traditional constructs that convert plural preferences over options into a complete ordering over sets of options. Each preference relation R_j is represented by a utility function u_j; and utility levels from the various functions are aggregated up using a probability vector $p = (p_1,\ldots,p_m)$. The order of events coincides with the effective freedom model in which the agent first decides between opportunity sets and then, after the preference ordering is revealed, selects the best alternative from the chosen set. Let $u_j(A) = \max_{a\Sigma A} u_j(a)$ denote the indirect utility function that extends u_j to sets of alternatives by associating with each set the utility level of its best element. The *expected indirect utility* of a set is defined as $U_p(A) = p_1u_1(A)+\ldots+p_mu_m(A)$; this is the numerical measure of opportunity freedom employed by Arrow (1995). We will denote associated freedom ordering by R_p.

Since each R_j is represented by u_j, it follows that the relation R* has a vector valued representation using $u(A) = (u_1(A),\ldots,u_m(A))$, the vector of Arrow's indirect utilities. It is immediately clear that so long as p assigns positive probability $p_j > 0$ to each u_j, the resulting freedom ordering R_p is an extension of the effective freedom relation R* (and hence R* is a subrelation of R_p). Moreover, the freedom measure $U_p(A) = pu(A)$ partially represents R*.

To illustrate how the Arrow approach expands the comparisons offered by R*, let us return to the utility version of the Bollywood example and set p = (1/2,1/2). Since u({w,y,z}) = u({w,z}) = (5,3), it follows that U_p({w,y,z}) = U_p({w,z}) = 4; both {w,y,z} I*{w,z} and {w,y,z} I_p {w,z} hold, as we would expect, since y would never be selected under either realization. As for the singleton sets, the indirect utility vectors associated with {w}, {y}, and {z} are, respectively, (5,-5), (0,0) and (-3,3), and are not ranked by R*; the freedom measure U_p takes a value of 0 for each, and so {w} I_p {y} I_p {z}. Finally, adding an alternative to a singleton set always raises the effective utility and the freedom measure in this example. For instance u({w,y}) = (5,0) > (5,-5) = u({w}), while U_p({w,y}) = 2.5 > 0 = U_p({w}), and so both rankings are in agreement.

Turning to the properties satisfied this relation, Arrow (1995) noted that R_p is a complete ordering that satisfies monotonicity. In general, though, R_p does not satisfy any of the other properties discussed above. R_p generally violates the Pattanaik and Xu (1990) properties since its subrelation R* can violate them. However, it may be less obvious why the properties satisfied by R* also can be violated by R_p, so we will reexamine the Bollywood example to illustrate why this occurs. Let A = {z}, B = {w}, and C = {y} so that A R_p B and the other requirements of simple semi independence hold. Now consider A∪C.

When the utility is known to be u_1, the option y would be chosen yielding a utility of 0; when utility is u_2 the choice would be z and the utility would be 3. Consequently, $u(A \cup C) = (0,3)$ which leads to $U_p(A \cup C) = 1.5$. In contrast, $B \cup C$ has the best element w for utility u_1 and the best element y for utility u_2, yielding $u(B \cup C) = (5,0)$ and hence $U_p(B \cup C) = 2.5 > U_p(A \cup C)$. This goes against the conclusions of simple semi-independence. For simple composition, select A = {y}, B = {w}, and D = {z}. Since all singleton sets have expected indirect utility of 0 it follows that A R_p B, A R_p D, and all the other conditions for simple composition are met. And yet $U_p(B \cup D) = 4 > U_p(A)$ and hence $B \cup D$ P_p A, violating the conclusions of simple composition. In this example, we see how Arrow's requirement of completeness has forced a ranking (e.g., indifference) among singleton opportunity sets that have very different preference-contingent utility levels and hence are unranked by R*. With this key information aggregated away it is impossible to predict the combined value of a set of alternative, as this depends on the precise way that the utility vectors match up.[30] As a result the intuitive composition property satisfied by R* is violated by R_p.

At the same time, the structure underlying R_p agrees with the constructs standardly used in modeling an agent facing some form of uncertainty. The resulting "expected indirect utility" freedom measure carries with it a significant amount of meaning and is readily computed, so long as the utility representations and probability vector underlying it can be agreed upon.[31] In addition, the structure itself can represent a surprisingly wide range of conceptions of freedom. At one extreme, when there is a single preference ordering in $\mathfrak{R}$ (or the probability vector over plural preferences is degenerate so that a single preference ordering has all the weight), R_p becomes the indirect utility freedom ordering associated with that preference ordering. Alternatively, suppose that each preference relation is represented by $u_j(x) = |L_j(x)|$, so that utility is the number of elements in the lower contour set of x. Suppose further that $\mathfrak{R}$ is the set of all logically possible preference orderings and each is considered to be equally likely by p. If opportunity sets A and B satisfy $|A| = |B|$, then by the symmetry of the scenario, their indirect utility vectors must be permutations of one another and, hence, $U_p(A) = U_p(B)$. Moreover, if $x \sum X$ is an option that is not in A, then by the assumption on preferences A and {x} cannot be ranked by R* and so $A \cup \{x\}$ P* A. This means that $u(A \cup \{x\}) > u(A)$ and thus $U_p(A \cup \{x\}) > U_p(A)$. It follows that R_p is simply the cardinality ranking R^C. Hence, Pattanaik and Xu's (1990) freedom ranking R^C can arise in the Arrow framework when the decision maker has maximal uncertainty about preferences (in that all logically possible preferences are equally probable) and utilities take a particularly simple form (the number of weakly dominated alternatives). This shows both the flexibility of the Arrow framework and the unrealistic assumptions that are implicit in the cardinal freedom ranking.

V.4. Reflections on Plural Preferences and Effective Freedom

When a decision maker has several preference orderings, any one of which *may* become relevant for a choice situation, this can create value for certain options that, in the course of events, were not chosen. This view of freedom (as the range of options than might have value in potential choice situations) relies upon multiple futures to construct value today. If resolution of all uncertainty occurs up front, leading to the single preference model and the indirect utility ranking, the value from unchosen options is lost. If the resolution is in the distant future, with all actions being taken under the same cloud of uncertainty, then the ability to respond and make particular use of valuable options is muted. The two serial choice problems (namely, set-choice and element-choice) becomes a single and essentially static decision problem, and the focus once again is the chosen elements (not the unchosen). In contrast, the effective

[30] The negative correlation of u_1 and u_2 would appear to have played a pivotal role in the above example.

[31] The fact that the approach requires agreement on these elements can be a drawback in practical applications.

freedom model embodies the perspective of "freedom as flexibility" – the ability to react and do well in the face of changing circumstances.

If uncertainty and preferences can be quantified into probabilities and utilities, the toolkit of expected utility can be brought to bear on the issue, resulting in a precise freedom measure and complete ordering of freedom. However, now it is the robustness of the measure (and the comparisons it makes) that is in doubt. As a remedy we might begin with a *set* of possible probability vectors and inquire whether the original comparison obtained with p would be preserved *for all probability vectors* in the set. This is analogous to Bewley's (2002) model of Knightian uncertainty, and several other related constructs for modeling ambiguity.[32] Various shapes of sets might be entertained, but one interesting case where the set is the entire simplex of probability vectors, and where *full robustness* is being required. This may be viewed as a case of maximal ambiguity, and where one's confidence in the initial probability vector is so small, one feels compelled to check every possible alternative. It is easy to see that Arrovian comparisons of freedom are fully robust to changes in the probability vector exactly when the effective freedom ranking R^* holds. Thus, Arrow's ranking R_p and R^* are two extremes of a model with varying levels of ambiguity: with the complete ordering R_p arising when there is no ambiguity concerning p; and the quasiordering R^* arising when ambiguity concerning p is largest.

VI. Conclusions

The traditional model of economic decision making leaves little room for valuing options that are not chosen by the decision maker. Yet as noted by Arrow (1995), the idea that increased flexibility – "keeping your options open" – can have value in a variety of contexts has been fully understood by economists for many years time as evidenced by the various papers he cites in the literature. The recent work on evaluating opportunity freedom has been exploring related insight in the context of normative economics, with the ultimate goal of enriching the basis for evaluating wellbeing and providing practical measurement tools for the capability approach.

In this paper, we have reviewed three main approaches to evaluating and measuring opportunity freedom. The first takes the untenable position that there is simply no way of determining the relative desirability of options. Left with a minimal informational base for evaluation, the cardinal approach equates greater freedom with greater numbers of options. The Pattanaik and Xu (1990) characterization of the cardinal ranking formally captures this lack of information in its anonymity axiom, that requires all singleton sets to have equal levels of opportunity freedom.

The second approach brings into the discussion exactly the sort of preference information needed to evaluate the quality of the various options. But once the agent's preference information has been introduced into the model, it is natural to ignore all but the very best option in a set (according to the fixed and known preferences) and hence to use an indirect utility approach – which equates opportunity freedom with the level of utility of the chosen alternative, and ignoring the other "unchosen" alternatives. The indirect utility approach to freedom likewise can be characterized with the key axiom "extending" the ordering of alternatives to the freedom ordering of singleton sets. Using the two building blocks of cardinal freedom (representing a concern for quantity) and indirect utility freedom (embodying the concern for quality), many hybrid measures of freedom can be constructed.

The third approach to evaluating freedom assumes the decision maker has multiple preference orderings, each of which might well be the relevant one when an alternative is to be selected from an opportunity

[32] See also Gilboa (2009) and Foster, Seth, and McGillivray (2009).

set. Consequently, each element is viewed as a bundle of possible utility levels, and a set is seen as a bundle of maximum achievable (or indirect) utility levels, one for each realized preference. The effective freedom quasiordering is vector dominance over these contingent utility levels; in other words, one set has as much effective freedom as another if for each preference relation, it has at least as good an option as the second set, according to that preference. A set's effective freedom is unchanged when a dominated option is added to the set. However, the effective freedom level rises if the added option is strictly better than the options in A for at lease one of the possible preferences. The effective freedom can be represented in several ways and a subrelation and an extension have been constructed to fit alternative models of choice. The taxonomy of the paper was constructed to readily allow a comparison of properties and characteristics of the various freedom rankings, and to reveal the hidden relationships among the different approaches. It is hoped that this structure will help future researchers formulate practical ways of measuring the freedoms that constitute our wellbeing.

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